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Uniform Asymptotic Stability via the Limiting Equations

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1. INTRODUCTION

The construction of Liapunov functions is a most powerful tool for establishing asymptotic and uniform asymptotic stability in ordinary differential equations; see [11]. One finds, however, that in many cases it is very complicated to construct the appropriate Liapunov function. Another powerful tool was developed by LaSalle: by combining information obtained from simple and natural Liapunov functions with information about geometric and invariance properties of limit sets, one can establish asymptotic properties of solutions including asymptotic stability; see [6, 7]. The LaSalle principle enables us to handle a variety of equations for which the geometry of the law of motion is detectable, and then relatively simple Liapunov functions are sufficient. (For the theory of nonautonomous equations see [4], a survey.) The two methods are direct methods; i.e., one should be able to discover the asymptotic stability by looking at the equation and without, for instance, computing the solutions.

The purpose of the present paper is to push LaSalle's ideas further, to include uniform asymptotic stability. We again insist on direct methods. We are willing to use the geometry and asymptotic properties of the equation, but would like to relax as much as possible the demands on the Liapunov functions. A major role in our techniques is played by the limiting equations—the equations which describe the limiting behavior of the original nonautonomous law of motion; see [4, 6, 9]. Our main result is the characterization (under certain conditions) of uniform asymptotic stability with respect to an equation by mere asymptotic stability, but with respect to all its limiting equations. This abstract characterization becomes practical in those cases where the structure of the limiting equations is relatively clear. We can use the LaSalle principle for handling the limiting equations and therefore obtain the results for the original equation. We will present some examples below.

The paper is organized as follows. The main results are presented in Section 3. Necessary preliminaries for applying the results are listed in Section 4. An example is given in Section 5 in which we analyze a damped harmonic oscillator.

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Another application is treated in Section 6, in which we generalize some results of [8] to nonlinear systems. In Sections 7 and 8 we examine the changes in the main results that occur if two conditions are respectively dropped. Two more applications are given in the final sections. In Section 9 a remark on almost periodic equations is made and in Section 10 a perturbation result is discussed.

Throughout the paper we work under two restrictions which are made primarily for simplicity of presentation. We always deal with the asymptotic stability of the origin. The generalization to compact sets is straightforward. We also discuss the stability properties only on the positive half line, and therefore use only positive limiting equations. By introducing negative limiting equations the stability on the whole line can be handled as well.

2. Notations and Assumptions

We consider nonautonomous equations of the form

$$\dot{x} = f(x, t), \tag{$*$}$$

where $x \in R^n$, the n-dimensional Euclidean space; and f: $R^n \times R_+ \to R^n$, where $R_+ = \{t: t \geqslant 0\}$. We always assume that f is continuous in x, is measurable in t, and satisfies the Caratheodory conditions locally (i.e., for x in bounded sets $|f(x, t)| \leqslant h(t)$ with h locally integrable) and that the following assumption on the global behavior of f holds.

Assumption (A). *For each compact set $K \subset R^n$ there is a nondecreasing function μ_K: $R_+ \to R_+$, continuous at 0 with $\mu_K(0) = 0$, such that if u: $[\alpha, \beta] \to K$ is continuous, then*

$$\left| \int_\alpha^\beta f(u(s), s)\, ds \right| \leqslant \mu_K(\beta - \alpha);$$

in particular the integral exists, and μ_K is a modulus of continuity for the indefinite integral.

The assumption is fairly relaxed; compare [3]. Unless otherwise stated, we do not assume uniqueness of solutions of the initial-value problems associated with $(*)$.

The $\dot{x}$ above means derivation with respect to time, $|x|$ will denote a norm of x and $x \cdot y$ denotes the scalar product of x and y. Integration is taken in the Lebesgue sense, equalities of functions of time are meant almost everywhere. The 0 denotes both the number zero and the origin (the zero element) of R^n.

3. The Main Results

We shall first state our main results and only then define and discuss all the terms in the statements. An earlier version of the "only if" parts of the statements below can be found in Sell [9].

Theorem A. *Let*

$$\dot{x} = f(x, t) \tag{$*$}$$

be positively precompact and regular. Suppose that 0 *is uniformly stable with respect to* $(*)$. *Then a set* W *is a region of uniform attraction of* 0 *with respect to* $(*)$ *if and only if* W *is a region of attraction of* 0 *with respect to every limiting equation of* $(*)$.

The following two theorems are actually corollaries of the previous theorem (and of course the definitions below).

Theorem B. *Under the conditions of Theorem* A, 0 *is uniformly asymptotically stable with respect to* $(*)$ *if and only if there exists a neighborhood* W *of* 0 *such that* W *is a region of attraction of* 0 *with respect to every limiting equation of* $(*)$.

Theorem C. *Under the conditions of Theorem* A, 0 *is globally uniformly asymptotically stable with respect to* $(*)$, *if and only if* 0 *is globally asymptotically stable with respect to every limiting equation of* $(*)$.

We shall see by means of a counterexample that it is not true that the uniform asymptotic stability of 0 with respect to $(*)$ is equivalent to the asymptotic stability of 0 with respect to every limiting equation of $(*)$.

The concepts of stability that we use are standard, but since there is no agreement in the literature on terminology we shall give precise definitions here. The origin 0 is *uniformly stable* with respect to an equation

$$\dot{x} = g(x, t) \tag{1}$$

if for every $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that if φ is a solution of (1) and $|\varphi(t_0)| \leqslant \delta$ for a certain t_0 then $|\varphi(t)| \leqslant \epsilon$ for all $t \geqslant t_0$. (If $\delta = \delta(\epsilon, t_0)$ we have *stability*.) A neighborhood W of 0 is a *region of attraction* of 0 with respect to (1) if whenever φ is a solution of (1) and $\varphi(t_0) \in W$ for a certain t_0 then $\varphi(t) \to 0$ as $t \to \infty$. The neighborhood W is a *region of uniform attraction* if the convergence to 0 above is uniform in the initial time t_0, i.e., for every compact $K \subset W$ and every $\epsilon > 0$ a positive T exists such that whenever a solution φ of (1) satisfies $\varphi(t_0) \in K$ then $|\varphi(t)| \leqslant \epsilon$ for $t \geqslant t_0 + T$. The origin 0 is *uniformly asymptotically stable* with respect to (1) if it is uniformly stable and there exists a neighborhood of 0 which is a region of uniform attraction of 0 (if the neighborhood is only a region of attraction we have *asymptotic* *stability*). The origin is *globally uniformly asymptotically stable* if it is uniformly stable and R^n is a region of uniform attraction of 0.

We shall consider below the stability properties of solutions of equations which are not necessarily ordinary differential equations. All the above concepts of stability are valid with no change when (1) is replaced by a more general type of equation.

An ordinary differential equation $\dot{x} = g(x, t)$ is a *limiting equation* of

$$\dot{x} = f(x, t) \tag{$*$}$$

if there is a sequence $t_k \to \infty$ such that whenever φ_k: $[a, b] \to R^n$ is a sequence of continuous functions which converge uniformly to φ: $[a, b] \to R^n$ then

$$\int_a^b g(\varphi(s), s)\, ds = \lim_{k\to\infty} \int_a^b f(\varphi_k(s), t_k + s)\, ds. \tag{3.1}$$

This convergence concept is fairly weak, and covers a wide family of functions. In many cases the convergence has a nicer representation; a useful case will be described in Section 4 below.

We denote by f^τ the translation of f by the time τ, i.e., $f^\tau(x, s) = f(x, \tau + s)$. If (3.1) holds we say that f^{t_k} *converges* to g; compare [4].

We now make things more complicated by considering limiting equations which are not ordinary differential equations. References [2, 3] are devoted to the study of such limiting equations. In short, we consider equations determined by an operator H (an ordinary integral-like operator) which associates with each R^n-valued continuous function φ with domain of definition (α, β) and with each τ in (α, β) a continuous function $H_\tau\varphi$ from (α, β) into R^n such that

(a) H_τ is continuous in the sup norm on compact intervals,

(b) $(H_\tau\varphi)(t) = (H_\tau\varphi)(s) + (H_s\varphi)(t)$ for all s, t and τ in the domain of φ.

Consistency with Assumption (A) is also required by asking that whenever φ: $[a, b] \to K$ is continuous then $H_a\varphi$ admits μ_K as a modulus of continuity on $[a, b]$. A function ψ is a *solution* of the operator equation $x = Hx$ if $\psi(t) = \psi(\tau) + (H_\tau\psi)(t)$. Maximally defined solutions of the initial-value problem $x = Hx$, $x(t_0) = x_0$ exist; see [3].

An operator equation $x = Hx$ is a *limiting equation* of $(*)$ if there exists a sequence $t_k \to \infty$ such that whenever φ_k: $[a, b] \to R^n$ is a sequence of continuous functions which converge uniformly to φ: $[a, b] \to R^n$ then

$$(H_a\varphi)(b) = \lim_{k\to\infty} \int_a^b f(\varphi_k(s), t_k + s)\, ds.$$

We then say that f^{t_k} *converge* to H.

It should be clear that an ordinary differential equation $\dot{x} = g(x, t)$ has a representation as an operator equation of the above type by letting $(H_a\varphi)(b) = \int_a^b g(\varphi(s), s)\, ds$. But not every operator equation can be obtained in this way from an ordinary differential equation.

Equation $(*)$ is *positively precompact* if for any sequence $t_j \to \infty$ there exist a subsequence t_k and an operator equation $x = Hx$ such that f^{t_k} converge to H as $k \to \infty$; in particular, $x = Hx$ is a limiting equation. Equation $(*)$ is positively precompact in the *restricted sense* if it is positively precompact and all the limiting equations are ordinary differential equations. (Note that our theorems are stated for merely positively precompact equations; for computational purposes it will be convenient to have the precompactness in the restricted sense.)

Equation $(*)$ is *regular* if for every limiting equation $x = Hx$ of $(*)$ and every pair (t_0, x_0) the initial-value problem $x = Hx$, $x(t_0) = x_0$ has a unique solution. (In this terminology we basically follow Sell [9], but note that Sell demands that the uniqueness hold also for the original equation; also, the convergence in [9] is a stronger convergence and all the limiting equations are ordinary differential equations.)

Proof of Theorem A. We shall utilize the following continuous-dependence result: Let f^{t_k} converge to H and let $x_k \to x_0$. If φ_k is a solution of the initial-value problem $\dot{x} = f^{t_k}(x, t)$, $x(t_0) = x_k$ and if for each k the solution φ_k is defined over $[t_0, t_1]$, then φ_k converge, as $k \to \infty$, uniformly on $[t_0, t_1]$ to the solution φ of $x = Hx$, $x(t_0) = x_0$ (the uniqueness of the latter is implied by the regularity). This continuous-dependence result is not hard to prove and it also follows from [3, Theorem 5.3]. As an immediate consequence it follows that the uniform stability of 0 with respect to $(*)$ implies that 0 is uniformly stable with respect to every limiting equation of $(*)$, and furthermore, the estimates $\delta(\epsilon)$ in the definition of uniform stability are inherited by the limiting equations.

Suppose that W is a region of uniform attraction of 0 with respect to $(*)$. Let $K \subset W$ be compact and let $\epsilon > 0$. Let $T = T(K, \epsilon)$ be the estimate for the uniform attractivity with respect to $(*)$. Let $x = Hx$ be a limiting equation. we have to show that the solution φ of $x = Hx$, $x(t_0) = x_0$ converges to 0 if $x_0 \in K$. There exists a sequence $t_k \to \infty$ such that f^{t_k} converge to H. By the continuous-dependence result φ is the uniform limit on compact intervals of the solutions φ_k of $\dot{x} = f^{t_k}(x, t)$, $x(t_0) = x_0$. Define ψ_k by $\psi_k(t) = \varphi_k(t - t_k)$. Then ψ_k is a solution of $(*)$ with $\psi_k(t_k + t_0) = x_0$. Therefore the uniform attractivity of 0 implies that $|\psi_k(t_k + t_0 + t)| = |\varphi_k(t_0 + t)| \leqslant \epsilon$ if $t \geqslant T$. The inequality is maintained by the limit and thus $|\varphi(t_0 + t)| \leqslant \epsilon$ if $t \geqslant T$. This completes the "only if" part of the theorem.

Suppose now that W is not a region of uniform attraction of 0 with respect to $(*)$. Then there exist: a compact set $K \subset W$, a positive ϵ, a (without loss of generality, converging) sequence $x_k \to x_0$ in K, a sequence t_k in R_+, and another sequence $T_k \to \infty$ such that for each k a solution ψ_k of $(*)$ exists with $\psi_k(t_k) = x_k$ and $|\psi_k(t_k + T_k)| \geqslant \epsilon$. Let $\delta = \delta(\epsilon/2)$ be given by the uniform stability; then clearly $|\psi_k(\tau)| \geqslant \delta$ for $t_k \leqslant \tau \leqslant t_k + T_k$. The positive precompactness of $(*)$ implies that for a subsequence of t_k (and without loss of generality let it be the sequence itself), f^{t_k} converge to a limiting operator H. Define φ_k by $\varphi_k(t) = \psi_k(t_k + t)$. Then φ_k is a solution of $\dot{x} = f^{t_k}(x, t)$, $x(0) = x_k$. By the continuous-dependence result φ_k converge uniformly on compact intervals to the solution φ of $\dot{x} = Hx$, $x(0) = x_0$. The inequalities $|\varphi_k(\tau)| \geqslant \delta$ for $0 \leqslant \tau \leqslant T_k$ imply that $|\varphi(\tau)| \geqslant \delta$ for every $\tau \geqslant 0$, and in particular $\varphi(t)$ does not converge to 0 as $t \to \infty$. Therefore W is not a region of attraction of 0 with respect to $\dot{x} = Hx$. This proves the "if" part and completes the proof of the theorem.

Remark 3.1. Notice that in the first part we have actually proved that W is a region of uniform attraction for every limiting equation $\dot{x} = Hx$, and with the same estimates $T(K, \epsilon)$ as for $(*)$. Moreover, the uniform attractivity of 0 with respect to $(*)$ was given on R_+, while the limiting equations can be naturally extended to the whole time line R, and 0 is uniformly stable and a uniform attractor with respect to them on all R.

It might happen (even under the conditions of Theorem A) that 0 is uniformly asymptotically stable with respect to every limiting equation of $(*)$, without being uniformly asymptotically stable with respect to $(*)$ itself. The point is, of course, that each limiting equation has its own region of uniform attraction to 0, but there is no common region for all the limiting equations:

EXAMPLE. For each $\eta = 1, 2, 3,\ldots$, let f_η: $R \to R$ be defined by $f_\eta(x) = -x \cdot \min(1, |2 - \eta |x||)$. Consider now the sequence of intervals $[2^k, 2^{k+1})$. Let $f(x, t)$ be defined as follows: For t in a fixed interval $f(x, t) = f_\eta(x)$ for a certain fixed η, and such that each η is used on an infinite number of intervals. The constant function $\varphi(t) = 2\eta^{-1}$ is a solution on $[2^k, 2^{k+1})$ of $\dot{x} = f(x, t)$ if $f(x, t) = f_\eta(x)$ on this interval; therefore 0 is not uniformly asymptotically stable with respect to $\dot{x} = f(x, t)$. (It is, however, uniformly stable since $\operatorname{sgn} \dot{x} = -\operatorname{sgn} x$.) The family of limiting equations can be easily computed. Every limiting equation is an ordinary differential equation $\dot{x} = g(x, t)$ satisfying either $g(x, t) = f_\eta(x)$ for t greater than a certain t_0 and for a fixed η, or $g(x, t) = -x$ for t greater than a certain t_0. In each case 0 is uniformly asymptotically stable with respect to the limiting equation.

4. PRELIMINARIES

We shall first provide conditions that will guarantee that the positive precompactness and regularity assumptions in the main results do hold. Results from

[1–3] will be quoted, but we feel that there is still room for improvement. We shall also recall the invariance principle for nonautonomous equations. It will serve us extensively in the examples and applications below.

PROPOSITION 4.1 (see [3, Theorem 8.1]). *Suppose that for every fixed compact* $K \subset R^n$ *the function* f *satisfies* $|f(x, s) - f(y, s)| \leqslant \nu_K(|x - y|, s)$ *whenever* $x, y \in K$, *where* $\nu_K(r, s)$ *is nondecreasing in* r *and continuous in* r *at* $r = 0$, *and* $\nu_K(0, s) = 0$. *Also assume that* $\nu_K(r, s)$ *is locally integrable in* s *and for every* s $\int_s^{s+1} \nu_K(r, \sigma)\, d\sigma \leqslant N_K(r)$ *and* $N_K(r) \to 0$ *as* $r \to 0$. *Then* $\dot{x} = f(x, t)$ *is positively precompact.*

A particular case of Proposition 4.1 is the case when $f(x, s)$ is continuous in x uniformly in s.

PROPOSITION 4.2 (see [2]). *Suppose that for every compact* $K \subset R^n$ *the function* f *satisfies* $|f(x, s) - f(y, s)| \leqslant m_K(s) \mid x - y \mid$ *whenever* $x, y \in K$, *where* m_K *is locally integrable and such that* $\int_s^{s+1} m_K(\sigma)\, d\sigma \leqslant M$ *for a fixed* $M < \infty$ *and for all* s. *Then* $\dot{x} = f(x, t)$ *is positively precompact and regular.*

Remark. Under the conditions of Proposition 4.2 the convergence to the limiting equations is a metric convergence; see [2].

PROPOSITION 4.3 (see [1]). *Suppose that in addition to the assumptions in Proposition* 4.2 *the function* f *satisfies* $|f(x, s)| \leqslant b_K(s)$ *whenever* $x \in K$, *where* b_K *is locally integrable and* $B(t) = \int_{t_0}^t b_K(s)\, ds$ *is uniformly continuous in* t. *Then* $\dot{x} = f(x, t)$ *is positively precompact in the restricted sense* (*i.e., all the limiting equations are ordinary differential equations*) *and regular.*

PROPOSITION 4.4 (see [1]). *Under the assumptions of Proposition* 4.3 *the convergence to the limiting equations is a metric convergence. Also,* f^{t_k} *converge to* g *if and only if for every fixed interval* $[s_0, s_1]$ *and every fixed* x *the sequence of functions* $f^{t_k}(x, s)\colon [s_0, s_1] \to R^n$ *converges in the weak* L_1*-topology to* $g(x, s)$. *Alternatively, for every fixed* x *the function* $g(x, s)$ *is the almost-everywhere derivative of* $\lim_{k\to\infty} \int_0^s f(x, t_k + \sigma)\, d\sigma$.

For the next statement we need the concept of the *ω-limit set* of a function φ; we denote the set by $\Omega(\varphi)$ and it is defined by $\Omega(\varphi) = \{z\colon$ there exists a sequence $t_k \to \infty$ such that $z = \lim \varphi(t_k)\}$.

PROPOSITION 4.5 (see [4, Sects. 7, 13] or [3, Sect. 7]). *Let* $\dot{x} = f(x, t)$ *be positively precompact. If* φ *is a solution and if* $z \in \Omega(\varphi)$, *there exist a limiting equation* $x = Hx$ *and a solution* ψ *of the equation such that* $\psi(0) = z$ *and* $\psi(t) \in \Omega(\varphi)$ *for all* t *in the domain of* ψ.

5. An Example

Consider the damped oscillator equation $\ddot{x} + h(t)\dot{x} + x = 0$, or rather its equivalent system

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -h(t)\,y - x.\end{aligned} \tag{51.}$$

Assume that $h(t) \geqslant 0$ and that the indefinite integral of h is uniformly continuous (i.e., $\int_a^b h(s)\,ds \leqslant \mu(b - a)$, where μ is continuous at 0 and $\mu(0) = 0$). Then (5.1) satisfies Assumption (A) and furthermore the conditions of Proposition 4.3 hold. From Proposition 4.4 we can deduce that all the limiting equations of (5.1) have the same form, namely,

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -g(t)\,y - x,\end{aligned} \tag{5.2}$$

where g satisfies $\int_0^t g(s)\,ds = \lim \int_0^t h(t_k + s)\,ds$ for a certain sequence $t_k \to \infty$.

We shall now use the natural Liapunov function $V(x, y) = (x^2/2) + (y^2/2)$, i.e., the total energy at the state (x, y). The time derivative of $V(t) = V(x(t), y(t))$ is equal to $-h(t)\,y^2(t)$ if $(x(t), y(t))$ is a solution of (5.1) and then $V(t)$ is nondecreasing. From this we easily get that 0 is uniformly stable with respect to (5.1).

Theorem 5.1. *The origin is uniformly asymptotically stable with respect to* (5.1) *if and only if the system*

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x\end{aligned} \tag{5.3}$$

is not a limiting equation of (5.1).

Proof. The "only if" part follows from Theorem B (or Theorem A) by the fact that no nontrivial solution of (5.3) approaches 0. For proving the converse assume that the uniform asymptotic stability does not hold; then by Theorem B a solution φ of a limiting equation, say (5.2), does not converge to 0 as $t \to \infty$. Since $V(t) = V(\varphi(t))$ is nonincreasing it follows that $\varphi(t)$ approaches a set $\Gamma = \{(x, y)\colon x^2 + y^2 = c\}$ as $t \to \infty$, and $c > 0$ is constant. By the invariance principle stated in Proposition 4.5, a limiting equation of (5.2) and a solution ψ exist such that $\psi(t) \in \Gamma$ for every $t \geqslant 0$. This limiting equation is also a limiting equation of (5.1). The only possibility that such a function ψ is a solution of an equation of the form $\dot{x} = y$, $\dot{y} = -g_1(t)y - x$ is that $g_1(t) = 0$ for almost every t. Therefore the limiting equation guaranteed by the invariance is (5.3). This completes the proof.

We shall now state the necessary and sufficient conditions of the previous result in terms of the coefficients $h(t)$.

THEOREM 5.2. *The origin is uniformly asymptotically stable with respect to* (5.1) *if and only if*

$$\liminf_{\substack{t_0 \to \infty \\ t \to \infty}} \int_{t_0}^{t_0+t} h(s)\, ds > 0. \tag{5.4}$$

Proof. System (5.3) is a limiting equation of (5.1) if and only if there is a sequence $t_k \to \infty$ such that $\int_{t_k}^{t_k+\tau} h(s)\, ds \to 0$ as $k \to \infty$ for each τ; and therefore if and only if (5.4) does not hold. The result follows now from Theorem 5.1.

6. AN EXTENSION OF RESULTS OR MORGAN AND NARENDA

Morgan and Narenda [8] prove the following interesting result.

THEOREM 6.1. *Consider the linear system*

$$\dot{x} = -P(t)x, \tag{6.1}$$

where for each t *the matrix* $P(t)$ *is symmetric and positive semidefinite. Then* 0 *is uniformly asymptotically stable with respect to* (6.1) *if and only if there exist numbers* $a > 0$ *and* b *such that*

$$\int_{t_0}^{t} | P(s)\, x |\, ds \geq a(t - t_0) + b \tag{6.2}$$

for every t_0, $t \geq t_0$ *and every unit vector* x.

In [8] it is assumed that $P(t)$ is bounded and piecewise continuous, but as was noted by LaSalle [7] the proof works also under Assumption (A). (Paper [8] contains other results, too, including conditions equivalent to (6.2) in the particular case (6.1).) Our purpose in this section is to extend Theorem 6.1. By using the methods developed above we will show in particular that it is not the linearity of (6.1) that makes the growth condition work (in [8], too, a "nonlinear" lemma is the basis for the proof).

Notice that the function $V(x) = x \cdot x$ is a Liapunov function of (6.1), i.e., $V(x(t))$ is nonincreasing along solutions (this is implied by the semidefiniteness of $P(t)$). Therefore we easily get that 0 is uniformly stable with respect to (6.1). Also, (6.1) is positively precompact and regular; compare Proposition 4.3.

The growth condition in the Morgan–Narenda result is necessary in a very general situation.

THEOREM 6.2. *Suppose that*

$$\dot{x} = f(x, t) \tag{$*$}$$

is positively precompact and regular, and that 0 *is uniformly stable with respect to* $(*)$. *Then* 0 *is uniformly asymptotically stable with respect to* $(*)$ *only if there exists a neighborhood* W *of* 0 *such that for every* $\delta > 0$ *there are numbers* $a > 0$ *and* b *such that*

$$\int_{t_0}^{t} |f(x, s)|\, ds \geqslant a(t - t_0) + b \tag{6.3}$$

for every t_0, $t \geqslant t_0$ *and every* $x \in W$ *such that* $|x| \geqslant \delta$.

Proof. Let W be a compact neighborhood which is a region of uniform attraction of 0 with respect to $(*)$. If for every $a > 0$ and b there is a vector $x \in W$ such that $|x| \geqslant \delta$, and times t_0 and $t \geqslant t_0$ such that $\int_{t_0}^{t} |f(x, s)|\, ds < a(t - t_0) + b$, then there exists a sequence x_k (and without loss of generality $x_k \to x_0$ in W) and sequences t_k and $T_k \to \infty$ such that

$$\int_{t_k}^{t_k+T_k} |f(x_k, s)|\, ds \to 0 \qquad \text{as} \quad k \to \infty. \tag{6.4}$$

We can assume that $t_k \to \infty$. The positive precompactness of $(*)$ implies that a subsequence of t_k exists, and without loss of generality let it be the sequence itself, such that f^{t_k} converges to a limiting operator H. This convergence implies (see Section 3) that $\varphi(t) = x_0$ is a solution of $x = Hx$. Therefore W is not a region of attraction with respect to $x = Hx$ and this contradicts Theorem A. This completes the proof.

We cannot hope that (6.3) will be a sufficient condition in the most general case (take, for instance, $\dot{x} = y$, $\dot{y} = -x$). The result below, although seemingly very special, is phrased to extract the geometrical idea behind Theorem 6.1, and to use it in nonlinear systems. For simplicity of presentation we shall work under conditions which are more restricted than those which could probably be used.

THEOREM 6.3. *Suppose that the assumptions of Proposition* 4.3 *hold for Eq.* $(*)$ $\dot{x} = f(x, t)$. *Let* V *be a smooth positive definite function in a neighborhood* W *of* 0 *(i.e.,* $V(0) = 0$, $V(x) > 0$ *if* $x \neq 0$*). Suppose that for every* $x \in W$, *a convex compact set* $K(x) \subset R^n$ *is given such that* $f(x, t) \in K(x)$ *for every* t *and such that* $y \in K(x)$ *implies that either* grad $V(x) \cdot y < 0$ *or* $y = 0$. *Then* 0 *is uniformly asymptotically stable with respect to* $(*)$ *if for each* $\delta > 0$ *there are numbers* $a > 0$ *and* b *such that*

$$\int_{t_0}^{t} |f(x, s)|\, ds \geqslant a(t - t_0) + b \tag{6.5}$$

for every $x \in W$ *with* $|x| \geqslant \delta$.

Proof. Since grad $V(x) \cdot f(x, t) \leqslant 0$ it follows that V is a Liapunov function in W (i.e., $V(x(t))$ is nonincreasing if $x(t)$ is a solution of $(*)$) and therefore 0 is uniformly stable with respect to $(*)$. We will show that W is a region of attraction of 0 with respect to every limiting equation of $(*)$, and by Theorem B this will complete the proof. By the representation results in Proposition 4.4 each limiting equation of $(*)$ has the form

$$\dot{x} = g(x, t) \tag{6.6}$$

with $g(x, t) \in K(x)$ for every t. Indeed, $g(x, s)$ for s in a bounded interval is the weak L_1-limit of a sequence of functions with values in $K(x)$; therefore $g(x, s)$ is the limit in the L_1-norm of a sequence of convex combinations of elements of the original sequence. Since $K(x)$ is convex it follows that $g(x, s) \in K(x)$. If φ is a solution of (6.6) near 0 then $V(\varphi(t))$ is nonincreasing, this by the condition grad $V(\varphi(t)) \cdot g(\varphi(t), t) \leqslant 0$. Thus $V(x)$ is constant on the ω-limit set $\Omega(\varphi)$. By the invariance result, Proposition 4.5, a limiting equation, say $\dot{x} = h(x, t)$, of (6.6) exists and a solution ψ of this limiting equation with values in $\Omega(\varphi)$. Since grad $V(\psi(t)) \cdot h(\psi(t), t)$ measures the rate of change of $V(\psi(t))$, which is zero, and since $h(x, t) \in K(x)$ it follows that $h(\psi(t), t) = 0$ for almost every t, which means that $\psi(t)$ is a constant function, say $\psi(t) = c$. Suppose that $c \neq 0$. Since $\dot{x} = h(x, t)$ is a limiting equation of $(*)$, too (which follows since the convergence is metrizable; see Proposition 4.4), it follows that for every N

$$\left| \int_{t_0}^{t_0+N} f(c, s)\, ds \right| \tag{6.7}$$

can be made as small as we want by the proper choice of t_0. The geometric requirement (namely, grad $V(x) \cdot f(x, s) = 0$ implies $f(x, s) = 0$) implies that if (6.7) is small then $\int_{t_0}^{t_0+N} | f(c, s)|\, ds$ is also small. This contradicts (6.5).

Remark. Theorem 6.1 for $P(t)$ bounded, say $| P(t)| \leqslant \alpha$, is a particular case of Theorem 6.3 by the choices $V(x) = x \cdot x$ and $K(x) = \{Px\colon P$ is symmetric, positive semidefinite and $| P | \leqslant \alpha\}$.

Remark. The reader has surely noticed that (5.4) and (6.3) are two forms of the same growth condition (the change in form was made for convenience in presentation). The arguments of Section 5 are, however, different from those of the present section. One could probably combine the two results into one, but the cost would be changes in the Liapunov functions that we have used, which are the natural functions; this cost we want to avoid (see the Introduction).

Remark. The reason that we should expect conditions expressed in terms of integrals on the right-hand sides of the equations (such as (5.4) or (6.3)) is that the limits of these integrals determine the limiting equations in terms of which our main results are given; see Section 3.

7. Dropping the Regularity

In practice it might be hard to check the regularity of an equation, i.e., to check that initial-value problems for the limiting equations have unique solutions. In this section we want to examine what happens to Theorem A, and consequently to the other results, if the regularity assumption is dropped. Notice that the "if" part of Theorem A remains unchanged.

Theorem D. *Let*

$$\dot{x} = f(x, t) \tag{$*$}$$

be positively precompact and suppose that 0 *is uniformly stable with respect to* $(*)$. *Then*

(i) *W is a region of uniform attraction of* 0 *with respect to* $(*)$ *if W is a region of attraction of* 0 *with respect to every limiting equation of* $(*)$, *and*

(ii) *W is a region of uniform attraction of* 0 *with respect to* $(*)$ *only if W is a region of weak attraction of* 0 *with respect to every limiting equation of* $(*)$.

Before proceeding with the proof we must define the new concept (new here, but it appears frequently in the literature). The neighborhood W is a region of *weak attraction* of 0 with respect to a certain equation, if for every initial condition $x(t_0) = x_0$, $x_0 \in W$, the equation has at least one solution φ such that $\varphi(t) \to 0$ as $t \to \infty$. (Warning: Some authors give *weak attraction* a different meaning.)

Proof of Theorem D. We can modify slightly the proof of Theorem A and cover the present result. First, instead of using the continuous-dependence result mentioned above, we will use the following version, which suits the equations without uniqueness of solutions: Let f^{t_k} converge to H and let $x_k \to x_0$. If φ_k is a solution of the initial-value problem $\dot{x} = f^{t_k}(x, t)$, $x(t_0) = x_k$, and if all φ_k are defined on $[t_0, t_1]$, then a subsequence of φ_k converges uniformly on $[t_0, t_1]$ to a solution φ of $x = Hx$, $x(t_0) = x_0$. (See [3, Theorem 5.3].) The part of the proof that establishes the "only if" part of Theorem A can be repeated now with the only change that we take a subsequence of φ_k which converges to a solution φ on every compact interval, and therefore this φ satisfies $\varphi(t) \to 0$ as $t \to \infty$ (there might of course be other solutions that do not converge). Similar changes in the "if" part of the proof of Theorem A will be sufficient for producing a proof for the "if" part of the present result.

We shall construct examples that show that under the assumptions of Theorem D the attractivity with respect to the limiting equations is not necessary, and that the weak attractivity is not sufficient.

Example 7.1. Consider the scalar equation

$$\dot{x} = -x(|1 - x|^{1/2}). \tag{7.1}$$

Then R is a region of weak attraction of 0 with respect to (7.1). Yet it is not a region of attraction since $\varphi(t) = 1$ is a solution. Equation (7.1) is the only limiting equation of the two systems

$$\dot{x} = -x(|1 - x|^{1/2} + (1/t^2)), \tag{7.2}$$

$$\dot{x} = -x(|1 - x|^{1/2} - (1/t^2)). \tag{7.3}$$

It is also clear that 0 is uniformly stable with respect to both (7.2) and (7.3). Since any solution of (7.2) with positive initial value is smaller than the corresponding minimal solution of (7.1) it follows that R is a region of uniform attraction with respect to (7.2), yet R is not a region of uniform attraction with respect to (7.1). Since the solution of (7.3) with initial condition $x(t_0) \geqslant 1$ is greater than the corresponding maximal solution of (7.1) it follows that R is not a region of attraction of 0 with respect to (7.3), yet R is a region of weak attraction with respect to (7.1).

Generalizations of Theorems B and C in the spirit of Theorem D can easily be made, and Example 7.1 can be modified to include these generalizations.

8. Dropping the Uniform Stability

Suppose that we are interested in attractivity properties of 0 with respect to

$$\dot{x} = f(x, t), \tag{$*$}$$

but uniform stability of 0 is not guaranteed. What remains then of Theorems A and D? In the proof of the necessity of the conditions we have not used the uniform stability at all. Therefore we have

Theorem E. *Suppose that $(*)$ is positively precompact. Then the neighborhood W of 0 is a region of uniform attraction with respect to $(*)$ only if W is a region of weak attraction with respect to every limiting equation of $(*)$. In particular, if $(*)$ is regular then the weak attraction is actually attraction.*

Notice that uniform stability in Theorems A and D implies that $\varphi(t) = 0$ is a solution of $(*)$, and indeed the unique solution through 0. This implication does not hold in general, and is not needed in Theorem E.

The sufficiency part of Theorem A does not hold in general without the assumption of uniform stability, as the following example demonstrates.

Example 8.1. Let the scalar equation

$$\dot{x} = g(x, t) \tag{8.1}$$

be defined as follows. Let $h_k(t)$ be defined on the time interval $[2^k, 2^k + \log k)$ by $h_k(t) = k^{-1} \exp(t - 2^k)$. On the same interval we define g by $g(x, t) = -x$ if $x \leqslant 0$, by $g(x, t) = x$ if $0 \leqslant x \leqslant h_k(t)$, and by $g(x, t) = 2h_k(t) - x$ if $x \geqslant h_k(t)$. This we do for $k = 1, 2,...,$ and on the rest of the real line we define $g(x, t) = -x$. The origin (although stable) is not uniformly stable with respect to (8.1) since $h_k(t)$ is a solution of (8.1), $h_k(2^k) = k^{-1}$ and $h_k(2^k + \log k) = 1$. However, each limiting equation $\dot{x} = h(x, t)$ of (8.1) has the property that $h(x, t) = -x$ for t large enough; therefore 0 is globally uniformly asymptotically stable with respect to each limiting equation.

In the example above the attractivity of 0 is not uniform with respect to all limiting equations. If this uniformity is assumed, more can be obtained.

We say that a neighborhood W of 0 is a region of uniform attraction of 0 *collectively* with respect to a family $\mathscr{G}$ of equations if for every compact $K \subset W$ and $\epsilon > 0$ there is a $T = T(K, \epsilon)$ such that whenever φ is a solution of an equation in $\mathscr{G}$ and $\varphi(t_0) \in K$ then $|\varphi(t)| < \epsilon$ for $t \geqslant t_0 + T$.

THEOREM F. *Suppose that* $(*)$ *is positively precompact and regular. Suppose that* $f(0, t) = 0$ *for every* t *and that* 0 *is the unique solution of* $(*)$ *with* $x(t) = 0$. *Then* 0 *is uniformly asymptotically stable with respect to* $(*)$ *if and only if there is a neighborhood* W *of* 0 *which is a region of uniform attraction collectively with respect to the family of limiting equations of* $(*)$.

Proof. The "only if" part was actually proved in Theorem A; see Remark 3.1. For the "if" part it will be sufficient to prove that 0 is uniformly stable with respect to $(*)$; the result then will follow from Theorem A. If 0 is not uniformly stable with respect to $(*)$ then an $\epsilon > 0$ exists and sequences t_k, $x_k \to 0$ and $T_k \geqslant 0$ exist such that for each k a solution φ_k of $(*)$ satisfies $\varphi(t_k) = x_k$ while $|\varphi(t_k + T_k)| = \epsilon$. The uniqueness assumption about 0 together with continuous dependence implies that $t_k + T_k \to \infty$, and without loss of generality assume $t_k \to \infty$. We now distinguish between two cases.

Case I. The sequence T_k is bounded. In this case let t_j be a subsequence of t_k such that f^{t_j} converges, say to H, and T_j converges, say to T_0. The continuous-dependence result mentioned in the proof of Theorem A implies that ψ_j, defined by $\psi_j(t) = \varphi_j(t_j + t)$, converge uniformly on compact sets to the solution ψ of $x = Hx$, $x(0) = 0$ and therefore $|\psi(T_0)| = \epsilon$. But this contradicts the regularity which implies that 0 is the unique solution of $x = Hx$, $x(0) = 0$ (the function 0 is a solution of the latter equation since $f(t, 0) = 0$ for all t).

Case II. The sequence T_k is unbounded, and without loss of generality assume that $|\varphi(t_k + \tau)| < \epsilon$ if $0 \leqslant \tau < T_k$. Let $T(B, \epsilon/2)$ be the estimate of the uniform attraction to 0 which is valid for all the limiting equations with $B = \{x: |x| \leqslant \epsilon\}$. (Without loss of generality B is included in the region W

of uniform attraction.) Let $s_k = t_k + T_k - T(B, \epsilon/2)$. Let s_j be a subsequence of s_k such that f^{s_j} converges, say to H, and $\varphi(s_j)$ converges, say to y_0. The continuous-dependence result implies that ψ_j, defined by $\psi_j(t) = \varphi_j(s_j + t)$, converge to the solution ψ of $x = Hx$, $x(0) = y_0$, uniformly on $[0, T(B, \epsilon/2)]$. Now $\psi(0) \in B$ but $|\psi(T(B, \epsilon/2))| = \epsilon$, a contradiction. This completes the proof.

9. A Remark on Almost-Periodic Equations

If for every x the function $q(x, t)$ is almost periodic, uniformly for x in bounded sets, then the structure of the limiting equations of

$$\dot{x} = q(x, t) \tag{9.1}$$

is more evident. Indeed, the right-hand side of every limiting equation $\dot{x} = g(x, t)$ of (9.1) is obtained as a pointwise limit, uniform for all t, of a sequence $q^{t_k}(x, t)$. The positive precompactness in the restricted sense is self-evident. Also, for every $\epsilon > 0$ and every bounded set B there is a number τ such that

$$|q(x, s) - g(x, \tau + s)| < \epsilon \tag{9.2}$$

for every $x \in B$ and all s. We shall use Theorem D to prove the following result (which can probably be proved differently in an easier fashion).

Theorem 9.1. *Suppose that* 0 *is uniformly stable with respect to* (9.1). *If* 0 *is asymptotically stable then* 0 *is uniformly asymptotically stable with respect to* (9.1).

Proof. If 0 is not uniformly asymptotically stable then by Theorem D, a limiting equation, say $\dot{x} = g(x, t)$, and a solution φ of the equation exist in the region of attraction of 0 with respect to (9.1), such that $\varphi(t)$ does not converge to 0 as $t \to \infty$. The uniform stability implies that $\varphi(t)$ stays away from 0. Let τ be such that (9.2) holds; then the translation of $\varphi(s)$ by τ (i.e., $\varphi(\tau + s)$) is close to a solution χ of (9.1) with $\chi(0) = \varphi(\tau)$ (by using classical continuous-dependence results). In particular, if T is large and ϵ is small, $\varphi(\tau + T)$ is close to 0; indeed $\chi(s) \to 0$ as $s \to \infty$ (and uniformly for initial values in bounded sets) by the asymptotic stability of 0 with respect to (9.1): a contradiction which completes the proof.

Conley and Miller [5] produced an example of a linear equation $\dot{x} = -f(t)x$, with almost periodic coefficients f, such that 0 is asymptotically stable but not uniformly asymptotically stable or even uniformly stable. Our Theorem 9.1 implies that the absence of uniform stability is essential. With regard to the Conley–Miller example it is easy to deduce from the definition of almost periodicity that there is no almost-periodic function $f(t)$ such that

(1) $\int_0^t f \to \infty$ as $t \to \infty$ (this corresponds to the asymptotic stability of 0 with respect to $\dot{x} = -f(t)x$);

(2) $\int_{t_0}^t f \geqslant C > -\infty$ (uniform stability) and not

(3) $\inf_{t_0} \int_{t_0}^{t_0+T} f \to \infty$ as $T \to \infty$ (uniform asymptotic stability).

10. A Result on Perturbations

A particular case of the convergence introduced in Section 3 is the convergence to 0. We shall use the terminology of [3] and say that $g(x, t)$ *integrally converges* to zero as $t \to \infty$ if whenever φ_k: $[a, b] \to R^n$ is a sequence of functions converging uniformly on $[a, b]$, then

$$\lim_{t\to\infty} \int_a^b g(\varphi_k(s), t + s)\, ds = 0.$$

This convergence to zero is fairly weak, for instance weaker than the concept of *mostly approaches zero* which was introduced by Strauss and Yorke [10], which requires that

$$\lim_{t\to\infty} \sup_{\varphi} \left| \int_a^b g(\varphi(s), t + s)\, ds \right| = 0$$

when the sup is taken over all continuous functions φ: $[a, b] \to B$, where a, b and the bounded set B are fixed. The two concepts above are implied, for instance, by the requirement $g(x, t) \to 0$ as $t \to \infty$, uniformly for x in bounded sets.

We now compare an equation

$$\dot{x} = f(x, t) \tag{$*$}$$

with its perturbation

$$\dot{x} = f(x, t) + g(x, t) \tag{P}$$

when g integrally converges to zero as $t \to \infty$. Clearly from the definition of the limiting equations it follows that $(*)$ and (P) share the same family of limiting equations and in particular $(*)$ is positively precompact or regular if and only if (P) is positively precompact or regular, respectively. From the characterizations of uniform asymptotic stability above we can deduce the following.

Theorem G. *Suppose that* $(*)$ *is positively precompact and regular. Suppose that* 0 *is uniformly asymptotically stable with respect to* $(*)$ *and* 0 *is stable with respect to* (P) *and that* g *integrally converges to zero as* $t \to \infty$. *Then* 0 *is uniformly asymptotically stable with respect to* (P).

Proof. The uniform asymptotic stability of 0 with respect to either (*) or (P) is characterized by the existence of a region of uniform attraction collectively with respect to the (same family of limiting equations; see Theorem F.

As an illustrative example we can take the forced and damped harmonic oscillator

$$\ddot{x} + h(t)\dot{x} + x = g(x, t). \tag{10.1}$$

If $h(t) \geq 0$, if g integrally converges to zero as $t \to \infty$, and if 0 is stable with respect to the perturbed system $\dot{x} = y$, $\dot{y} = -h(t)y - x - g(x, t)$, then condition (5.4) implies the uniform asymptotic stability of the origin with respect to (10.1).

It seems to me that the result in Theorem G cannot be easily deduced by the techniques that heavily use Liapunov functions. Compare [10].

Note added in proof. Three preprints closely related to our subject (listed as Refs. [12–14]) were brought to my attention after I completed this work. The papers of Morgan [12] and Haddock and Parrot [13] pursue the direction of [8] and obtain considerably better results via improving the conditions demanded from the Liapunov functions. The approach of Bondi *et al.* [14] is closer to our approach (although the details seem to be different); their main result is related to our Theorem F.

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